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STABILITY OF STEADY FLOWS OF PERFECT INCOMPRESSIBLE FLUID WITH A FREE BOUNDARY

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The criterion of stability of steady flow of a perfect incompressible fluid bounded by solid walls, indicated by Arnol'd [1, 2], is extended to the case when a part of the flow region boundary is free and subjected to surface tension.

The method used in [1, 2] is a variant of Liapunov's second method and is close to Chetaev's method of bunching integrals [3] which was applied in [4] for investigating the stability of motion of a solid body with a cavity partly filled with fluid in potential motion. The Helmholtz — Thomson theorem on vortex conservation implies the existence of invariant foliation (Helmholtz foliation) in the considered dynamic system. Critical points of the energy functional on the layers of such foliation correspond to steady flows and, if the critical point is a nondegenerate maximum or minimum, the respective steady flow is stable. This shows that the property of fixed sign of some quadratic functional, i.e. of the second variation of energy on a Helmholtz layer, is the criterion of stability.

The general criterion of stability is applicable to parallel flows over a solid bottom and to circular flows. It appears that a convex profile of a stable flow must monotonically increase from the solid wall to the free surface, while a concave one must monotonically decrease.

1. The dynamic system and invariant foliation. Let $D(t)$ be a region dependent on time t of the three-dimensional space R^3 with smooth boundary $\Gamma(t)$ filled with a perfect incompressible fluid subjected to the action of potential mass forces of potential $U(x)$. Boundary $\Gamma(t)$ consists of a stationary solid wall Γ_1 and of the free boundary $\Gamma_2(t)$, which do not intersect at any t. In this case the velocity $\mathbf{v}(x, t)$ and pressure $p(x, t)$ in region $D(t)$ satisfy at all t Euler's equations (in the Gromeka — Lamb form) and the boundary conditions

$$\partial \mathbf{v} / \partial t + \mathbf{v} \times \mathbf{r} - \operatorname{grad} q, \quad \mathbf{r} = \operatorname{rot} \mathbf{v}, \quad q = U + p + {}^1/_2\, v^2 \tag{1.1}$$
$$\operatorname{div} \mathbf{v} = 0$$
$$v_n(x, t) = \mathbf{v}\mathbf{n} = 0, \ x \in \Gamma_1, \ v_n(x, t) = \varkappa_n$$
$$x \in \Gamma_2(t), \quad p(x, t) = 2\sigma H(x, t)$$

where $\sigma \geqslant 0$ is the surface tension coefficient and H is the mean curvature of surface $\Gamma_2(t)$ at point x, expressed in terms of the principal curvature radii by the formula

$$H = {}^1/_2\, (1 / R_1 + 1 / R_2)$$

where R_1 and R_2 are assumed positive when directed inward region $D(t)$, n is the unit vector of the outward normal of the region boundary, and $\varkappa_n$ is the normal component of the velocity of a boundary point.

A theorem on the existence of solutions of the input boundary value problem for system (1. 1) is not known. We assume that equations that correspond to weak perturbations of stable steady flows exist at all $t \in [0, \infty)$.

We denote by M the set of pairs (D, v), where D is the region with a smooth boundary whose part is Γ_1 and v is a smooth (up to the boundary) vector field in D, with $v_n(x) = 0$ and $x \in \Gamma_1$ (here and in what follows the term "smooth" means infinitely differentiable). Equations (1. 1) define in M a dynamic system.

Extending the definitions in [1] we call the elements (D, v) and (D', v') from M irrotational if there exists a smooth mapping g of region D into region D' such that $g\Gamma_1 = \Gamma_1$ which retains a volume element, and such that for any closed contour γ in region D

$$\oint_{\gamma} \mathrm{v}\, dx = \oint_{g\gamma} \mathrm{v}'\, dx$$

The (Helmholtz) foliation in M, i.e. the subdivision into equivalence classes is thus specified: two elements belong to one and the same layer when and only when their vorticities are the same.

Theorem 1. If $\mathrm{v}(x, t)$ satisfies Eqs. (1. 1) in region $D(t)$ at all $t \in (-\delta, \delta)$, where $x(t)$ is the fluid particle trajectory, then $(D(0), \mathrm{v}(x, 0))$ and $(D(t), \mathrm{v}(x, t))$ have the same vorticity and the mapping g converts $x(0)$ into $x(t)$.

The Helmholtz — Thomson theorem on vortex conservation, as formulated in [1], shows that the foliation defined above is invariant for the dynamic system in M, determined by Euler's equations.

2. The first and second variations of elements from M along the Helmholtz layers. The one-parameter set $(D(\tau), \mathrm{v}(x, \tau))$ determinate for $|\tau| < \delta$ and such that 1) the set $\{(x, \tau): x \in \partial D(\tau), |\tau| < \delta\}$ is a smooth submanifold in $R^3 \times (-\delta, \delta)$ and 2) $\mathrm{u}(x, \tau) = (\mathrm{v}(x, \tau), 0)$ is a vector field in region $\{(x, \tau): x \in D(\tau), |\tau| < \delta\}$ smooth up to the boundary, will be called smooth curve in M.

If the smooth curve $(D(\tau), \mathrm{v}(x, \tau))$ is entirely contained in layer F of the invariant foliation, then, according to the definition of equal vorticity elements, there exists a one-parameter set of smooth mappings $g_\tau : D(0) \to D(\tau)$ which retain the volume element and are such that for any contour $\gamma \subset D(0)$

$$\oint_{\gamma} \mathrm{v}(x, 0)\, dx = \oint_{g_\tau \gamma} \mathrm{v}(x, \tau)\, dx \tag{2.1}$$

Let (D, v) be an element from M. According to Weyl's expansion [4] v can be represented in the form

$$\mathrm{v} = \mathrm{v}_1 + \mathrm{v}_2, \ \operatorname{div} \mathrm{v}_1 = \operatorname{div} \mathrm{v}_2 = 0, \ v_{1n}(x) = 0, \ x \in \partial D \tag{2.2}$$

$$\mathrm{v}_2 = \operatorname{grad} \alpha, \ v_{2n}(x) = v_n(x), \ x \in \partial D$$

Roughly speaking, v_1 is the rate of internal mixing of the fluid and v_2 relates to the flow induced by variation of the free surface.

Velocity variations along curve $(D(\tau), v(x, \tau))$ are, in conformity with (2.2), of the form

$$\delta v = \delta v_1 + \delta v_2, \quad \delta^2 v = \delta^2 v_1 + \delta^2 v_2$$

Let the set g_τ from (2.1) be determined by $g_\tau(x(0)) = x(\tau)$, where $x(\tau)$ is the solution of the system of ordinary differential equations

$$dx / d\tau = f(x, \tau)$$

where $f(x, \tau)$ is the smooth solenoidal vector field specified in some neighborhood of D at $|\tau| < \delta$ such that $\operatorname{div}_x f(x, \tau) = 0, f_n(x,\tau) = 0, x \in \Gamma_1$.

L e m m a 1. If the closed contour $\gamma \subset D$ and

$$\oint_{g_\tau^{-1}\gamma} v_1(x, 0)\, dx = \oint_\gamma v_1(x, \tau)\, dx$$

then

$$\oint_\gamma (v_1(x, \tau) - v_1(x, 0))\, dx = \tau \oint_\gamma f(x, 0) \times r(x)\, dx +$$
$$\frac{\tau^2}{2} \oint_\gamma (f(x, 0) \times \{f(x, 0), r(x)\} + \varphi(x) \times r(x))\, dx + O(\tau^3)$$

where $r(x) = \operatorname{rot} v_1(x, 0)$, and $\{f(x, 0), r(x)\}$ is the Poisson bracket of vector fields $f(x, 0), r(x)$ and $\varphi(x) = (\partial f(x, \tau) / \partial \tau)_{\tau=0}$. A similar lemma was proved in [1] for the case of autonomous vector fields. The proof here does not greatly differ from that in [1].

C o r o l l a r y. The variations of field v_1 are of the form

$$\delta v_1 = (\partial v_1(x, \tau) / \partial \tau)_{\tau=0} = f \times r + \operatorname{grad} \alpha_1 \qquad (2.3)$$
$$\delta^2 v_1 = (\partial^2 v_1(x, \tau) / \partial \tau^2)_{\tau=0} = {}^1/_2 [f \times \{f, r\} + \varphi \times r] + \operatorname{grad} \alpha_2 \qquad (2.4)$$
$$\operatorname{div} \delta v_1 = \operatorname{div} \delta^2 v_2 = 0$$
$$(d[v_1(g_\tau x, \tau)\, n(g_\tau x, \tau)] / d\tau)_{\tau=0} = 0$$
$$(d^2[v_1(g_\tau x, \tau)\, n(g_\tau x, \tau)] / d\tau^2)_{\tau=0} = 0$$

where $x \in \partial D$, $n(g_\tau x, \tau)$ is the unit vector of the outward normal to $\partial D(\tau)$ at point $g_\tau x$, and α_1 and α_2 are determined by the last three of Eqs. (2.4).

L e m m a 2.

$$\delta v_2 = (\partial v_2(x, \tau) / \partial \tau)_{\tau=0} = \operatorname{grad} \beta_1 \qquad (2.5)$$
$$\delta^2 v_2 = (\partial^2 v_2(x, \tau) / \partial \tau^2)_{\tau=0} = \operatorname{grad} \beta_2 \qquad (2.6)$$

where β_1 and β_2 are harmonic functions in D , for which boundary conditions are obtained by differentiating condition $v_n = \varkappa_n$ with respect to τ at the boundary and, then, setting $\tau = 0$.

Let $h_\tau : D \to D(\tau)$ $(|\tau| < \delta)$ be a one-parameter set of mappings that retain a volume element and are smooth over the totality of arguments (x, τ). We then call the normal vector field in ∂D

$$(\delta \mathbf{D})\ (x) = (dh_\tau x\ /\ d\tau)_{\tau=0}\mathbf{n}\ (x),\ x \in \partial D$$

the variation of the free surface form. The following lemma justifies this designation by showing that $\delta \mathbf{D}$ depends only on the variation of the form of region D due to deformation of h_τ, and $\delta \mathbf{D} = 0$ when for small τ $D(\tau) = D$.

L e m m a 3. Let the two deformations $h_\tau^1,\ h_\tau^2 : D \to D(\tau)$ be specified, then

$$[(dh_\tau^1 x\ /\ d\tau)_{\tau=0}]_n = [(dh_\tau^2 x\ /\ d\tau)_{\tau=0}]_n$$

P r o o f. Since the smooth surface $(\partial D(\tau), \tau)$ can be specified in the neighborhood of any of its points in space $R^3 \times R$ by the equation $F(x, \tau) = 0$, $|\mathrm{grad}_x F(x, \tau)| \neq 0$, hence

$$\frac{d}{d\tau} h_\tau^i x n (h_\tau^i x, \tau) = -\frac{\partial F(h_\tau^i x, \tau)/d\tau}{|\mathrm{grad}_x F(h_\tau^i x, \tau)|}, \quad i = 1, 2 \tag{2.7}$$

where $\mathbf{n}(h_\tau^i x, \tau)$ is the outward normal to $\partial D(\tau)$ at point $h_\tau^i x$, $i = 1, 2$. Setting in (2.7) $\tau = 0$ we obtain the proof of the lemma (see, also, [6]).

The total energy of fluid is equal to the sum of kinetic energy, the potential energy of the mass force field and the surface potential energy of capillary forces. It is represented by the functional E in M which is defined by the equality

$$E = \int_{D(t)} \left(\frac{1}{2}\mathbf{v}^2 + U\right) dx + \sigma \int_{\Gamma_2(t)} ds$$

3. The variational principle and energy variation. T h e o r e m 2. The solenoidal vector field $\mathbf{v}(x)$ represents the velocity of the steady fluid flow in region D then and only then when the energy E on the Helmholtz layer has a local extremum at point $(D, \mathbf{v})$.

P r o o f. Let us determine the first variation of E at point $(D, \mathbf{v})$. Using formulas for derivatives of integrals over a region [6] and a surface (see, e.g., [7, 8] which depend on the parameter, from (2.3) and (2.5) with allowance for (1.1) we obtain

$$\delta E = \frac{dE}{d\tau}\bigg|_{\tau=0} = \int_D \mathbf{v}\delta\mathbf{v}\, dx + \int_{\partial D} \left(\frac{\mathbf{v}^2}{2} + U + 2\sigma H\right) f_n\, dS = \tag{3.1}$$

$$-\int_D \mathbf{f}\,\mathrm{grad}\, q\, dx + \int_{\partial D} \left(\frac{\mathbf{v}^2}{2} + U + 2\sigma H\right) f_n\, dS =$$

$$\int_{\partial D} (2\sigma H - p) f_n\, dS = 0$$

Conversely, let $\delta E = 0$, then, setting $\mathbf{f}|_{\tau=0} \equiv 0$, we obtain

$$0 = \int\limits_D \mathrm{v}\,\mathrm{grad}\,\beta_1\,dx = \int\limits_{\partial D} v_n \beta_1 dS$$

which shows that $v_n = 0$ in ∂D.

For an arbitrary smooth solenoidal vector field f from (2.3), (2.5), and (3.1) we furthermore have

$$0 = \int\limits_D \mathbf{f}\left(\mathbf{r} \times \mathbf{v} + \mathrm{grad}\left(\frac{\mathbf{v}^2}{2} + U + 2\sigma H\right)dx\right.$$

which in accordance with Weyl's expansion implies the existence of function $p\ (x)$ such that

$$\mathbf{r} \times \mathbf{v} + \mathrm{grad}\ (\mathbf{v}^2 / 2 + U) = -\mathrm{grad}\ p,\ p\ (x) = 2\sigma H\ (x),\ x \in \partial D$$

The theorem is proved.

First of all we note that for normal variations N of the free surface Γ_2 the mean curvature variation is defined by formula

$$\delta H = (2K - 4H^2)\ N - \Delta N \tag{3.2}$$

where K is the product of principal curvatures, Δ^* is the Laplace — Beltrami operator of surface Γ_2 (see, e.g., [8]). We continue the mean curvature of $H\ (x,\ \tau)$ as a function defined on surface $(\partial D\ (\tau),\ \tau)$ $(|\tau| < \delta)$ in R^4 into some neighborhood of that surface in R^4. Taking into account that in (3.2) according to Lemma 3 $\mathrm{N} = f_n$ and using formulas for integration of integrals over a region and surface dependent on parameter τ, from (2.4) and (2.6) we obtain

$$2\delta^2 E = \int\limits_D I\,dx + \int\limits_{\partial D} (J_1 + J_2)\,dS \tag{3.3}$$

$$I = (f \times r + \mathrm{grad}\,\alpha_1)^2 + (\mathrm{grad}\,\beta_1)^2 + (v \times f)\{f, r\}$$

$$J_1 = (2v\ (f \times r + \mathrm{grad}\ (\alpha_1 + \beta_1)) + \mathrm{grad}\ (v^2 / 2 + U)\ f)f_n$$

$$J_2 = f_n dH_S\ (f) + \sigma\ ((\nabla f_n)^2 + (2K - 4H^2)\ (f_n)^2$$

Note that for $\mathrm{f}_n \equiv 0$ in ∂D and $\beta_1 = 0$ in D formula (3.3) coincides with formula (5.1) in [1] in the case of flow of fluid in a container with solid walls. Thus, a stable steady flow with a free surface remains stable when a solid wall is substituted for the free surface (the principle of hardening).

4. **Stability of symmetric flows.** Let the solid wall Γ_1 and the potential U be invariant to translations along the Ox_1 -axis. We shall consider such motions of the fluid for which the free boundary $\Gamma_2(t)$ and the velocity field are periodic with respect to x_1 and of fixed period l. The equations of motion then admit the integral

$$L_1 = \int\limits_0^l \int\limits_{S(x_1, t)} \mathbf{v}\mathbf{e}_1\,dx$$

where $S(x_1, t)$ is the cross section of region $D(t)$ cut by a plane orthogonal to Ox_1 passing through point $(x_1, 0, 0)$, and e_1, e_2, e_3 are coordinate unit vectors.

Similarly, if the solid wall Γ_1 and the potential U are invariant with respect to rotation about the Ox_3 -axis, Euler's equations (1.1) — (1.3) have

$$K_3 = \int\limits_{D(t)} (\mathbf{v} \times \mathbf{R}) \mathbf{e}_3 \, dx, \quad \mathbf{R} = (x_1, x_2, x_3)$$

as the integral of the momentum of motion.

Theorem 3. The functional L_1 on the Helmholtz layer has an extremum at point $(D, \mathbf{v})$ then and only then when $\mathbf{v}$ is the velocity of steady flow in region D invariant with respect to translations along the Ox_1 -axis.

Theorem 4. The functional K on the Helmoltz layer has an extremum at point $(D, \mathbf{v})$ then and only then when $\mathbf{v}$ is the velocity of steady flow in region D that is invariant with respect to rotation about the Ox_3 -axis.

The proof of these two theorems is the same as the proof of Theorem 2.

For the second variations of L_1 and K along the Helmholtz layer we have

$$2\delta^2 L_1 = \int\limits_0^l \int\limits_{S(x_1)} (\mathbf{e}_1 \times \mathbf{f}) \{\mathbf{f}, \mathbf{r}\} \, dx + \int\limits_0^l \int\limits_{\partial S(x_1)} [2\mathbf{e}_1 (\mathbf{f} \times \mathbf{r} + \operatorname{grad}(\alpha_1 + \beta_1)) + \operatorname{grad}(\mathbf{v}\mathbf{e}_1)\mathbf{f}] \, f_n \, dS$$

$$2\delta^2 K_3 = \int\limits_D (\mathbf{R} \times e_3 \times \mathbf{f}) \{\mathbf{f}, \mathbf{r}\} \, dx + \int\limits_{\partial D} [2(\mathbf{f} \times \mathbf{r} + \operatorname{grad}(\alpha_1 + \beta_1)) \mathbf{R}\mathbf{e}_3 + \operatorname{grad}((\mathbf{v} \times \mathbf{R}) \mathbf{e}_3) \mathbf{f}] \, f_n \, dS$$

We thus obtain, as in [1], the following sufficient condition of stability of steady flows of perfect incompressible fluid with a free boundary.

Theorem 5. If the linear combination of second variations

$$\mu_1 \delta^2 E + \mu_2 \delta^2 L_1 + \mu_3 \delta^2 K_3 \tag{4.1}$$

taken along the Helmoltz layer exists and is a fixed sign, quadratic form of $\mathbf{f}$ the steady flow in region D at velocity $\mathbf{v}$ is stable with respect to small finite perturbations of velocity $\mathbf{v}$, vortex $\mathbf{r}$, and of the free surface form.

5. Examples. Let us investigate the stability of some steady plane flows of a perfect incompressible fluid with a free boundary with respect to plane perturbations.

Note that for the velocity $\mathbf{v}$ of a steady plane flow and any plane vector field $\mathbf{f}$

$$\mathbf{r}(x) = (0, 0, \mathbf{r}(x)), \quad \delta\mathbf{r} = \{\mathbf{f}, \mathbf{r}\} = \frac{\partial \mathbf{r}}{\partial x_1} f_1 + \frac{\partial \mathbf{r}}{\partial x_2} f_2$$

1°. Stability of plane-parallel flows. Let us investigate the stability of a parallel flow whose velocity field in the horizontal layer $D = \{(x_1, x_2): -\infty < x_1 < \infty, 0 \leqslant x_2 \leqslant a\}$ is $\mathbf{v}(x) = (u(x_2), 0)$, with $\Gamma_1 = \{(x_1, 0): -\infty < x_1 < \infty\}$ representing the solid wall and $\Gamma_2 = \{(x_1, a): -\infty < x_1 < \infty\}$ the free boundary. The fluid layer is assumed to be subjected to the force of gravity of potential $U = gx_2$.

Setting in (4.1) $\mu_1 = 2, \mu_2 = -2u(a), \mu_3 = 0$ we obtain

$$2\delta^2 E - 2u(a)\delta^2 L_1 = \int_0^l\int_0^a \left[(\delta v)^2 + \frac{u(x_2) - u(a)}{u''(x_2)}(\delta r)^2\right] dx_1\, dx_2 + \tag{5.1}$$

$$\int_0^l \left[g(f_2)^2 + \sigma\left(\frac{\partial f_2}{\partial x_1}\right)^2\right] dx_1$$

For the quadratic form (5.1) to be positive definite it is necessary and sufficient that the condition

$$[u(x_2) - u(a)] / u''(x_2) > 0, \quad x \in [0, a) \tag{5.2}$$

is satisfied.

The theory of oscillation of solutions of second order differential equations shows that inequality (5.2) is satisfied only in two cases:

1) $u''(x_2) \geqslant 0, \quad u(x_2) \geqslant u(a)$, and

2) $u''(x_2) \leqslant 0, \quad u(x_2) \leqslant u(a)$.

In the first case function $u(x_2)$ is concave and monotonically decreases, while in the second it is convex and increases monotonically.

Let us consider the case of $a < 0$ which corresponds to the flow of fluid over a ceiling. If, as previously, we set in (4.1)$\mu_1 = 2, \mu_2 = -2u(a), \mu_3 = 0$, the flow is then stable when condition (5.2) is satisfied and in addition $\sigma > gl^2 / (4\pi^2)$.

2°. Stability of flows along concentric circles. Let us investigate purely rotational flows in ring $D = \{(x_1, x_2) : a \leqslant \rho \leqslant b\}$ $(\rho = \sqrt{x_1^2 + x_2^2}) : v_\rho = 0, \ v_\theta = v_\theta(\rho)$. Let $\Gamma_1 = \{(x_1, x_2): \rho = b\}$ and $\Gamma_2 = \{(x_1, x_2): \rho = a\}$ be the solid wall and the free boundary, respectively; $\omega(\rho) = v_\theta(\rho) / \rho$ be the fluid particle angular velocity, and $\psi(\rho)$ be the stream function of the basic flow. We set in (4.1) $\mu_1 = 2, \mu_2 = 0$, and $\mu_3 = -2\omega(a)$. As the result we obtain the stability condition of the form

$$(\text{grad}\,\psi - {}^1/_2\ \omega\, a\, \text{grad}\,\rho^2) / \text{grad}\, r \geqslant 0 \tag{5.3}$$

Condition (5.3) is exactly similar to condition (5.2) and determines similar profiles of stable flows. However in this case we take Γ_2 as the solid wall and Γ_1 as the free boundary.

If we set in (4.1) $\mu_1 = 2$, $\mu_2 = 0$, and $\mu_3 = -2\omega(b)$, we obtain a stable flow when (5.3) is satisfied and in addition $\sigma > b^3\omega^2(b)$.

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